Elastic Postbuckling Predictions of Plates Using Discrete Elements

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Theme

THIS paper describes a discrete-element analysis capability for predicting the geometrically nonlinear, large-deflection and postbuckling responses of elastic plates with initial curvatures. Both the initial and additional deflections may be of the same order as the plate thickness. The nonlinear stiffness equation is derived by using the minimum potential energy theorem. The postbuckling response is predicted step by step by a linear incremental approach. Basic procedures for initiating a postbuckling configuration, correlating the middle-surface loads with the transverse loads, treating the edge conditions prescribed for the middle-surface movement, and finding the distribution of membrane stress caused by a large deflection are provided.

Content

The basic assumptions underlying this development can be described by the following set of finite-displacement, strain-displacement relationships where the geometric nonlinearity and initial deflection are incorporated in the plate,

$$\epsilon_x = \partial u/\partial x + \tfrac{1}{2}[\partial(w + w_0)/\partial x]^2 - \tfrac{1}{2}(\partial w_0/\partial x)^2 - z(\partial^2 w/\partial x^2) \quad (1a)$$

$$\epsilon_y = \partial v/\partial y + \tfrac{1}{2}[\partial(w + w_0)/\partial y]^2 - \tfrac{1}{2}(\partial w_0/\partial y)^2 - z(\partial^2 w/\partial y^2) \quad (1b)$$

$$\gamma_{xy} = \partial v/\partial x + \partial u/\partial y + [\partial(w + w_0)/\partial x][\partial(w + w_0)/\partial y] - [(\partial w_0/\partial x)\partial w_0/\partial y] - 2z\partial^2 w/\partial x\partial y \quad (1c)$$

in which u and v are the membrane displacements; w is the transverse deflection; and subscript 0 denotes initial quantity. It is noted that the use of Eqs. (1) limits the validity of the formulation to cases where the slopes of the deformed geometry are small relative to undeformed geometry.

Making use of Eqs. (1), the strain energy for a deformed plate is expressed in terms of the geometry, the stiffness (membrane, flexure, and coupling), and the derivatives of displacements u, v, and w. According to the order of the displacements, the strain energy is divided into second-, third-, and fourth-order groups. After adding the potential of the applied external loads to the strain energy, the discretized form of the potential energy for the discrete element is obtained by making use of the assumed displacement functions for whatever plate element model is chosen. Performing the partial differentiation of the total potential energy with respect to each degree of freedom, the nonlinear stiffness matrix equation for a discrete element is obtained.

$$\{\mathbf{p}\} = [k]\{\mathbf{q}\} + [[n_0] + \tfrac{1}{2}[n_1] + \tfrac{1}{3}[n_2]]\{\mathbf{q} + \mathbf{q}_0\} \quad (2)$$

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where $\{\mathbf{p}\}$ and $\{\mathbf{q}\}$ are vectors of nodal loads and displacements, respectively; $[k]$ is the linear stiffness matrix; $[n_0]$, $[n_1]$, and $[n_2]$ are the zero-, first-, and second-order incremental stiffness matrices which are indirectly functions of zero-, first-, and second-order of $\{\mathbf{q} + \mathbf{q}_0\}$ gross displacement vectors, respectively. Eq. (2) is appropriate for a mathematical iterative procedure.

Performing a first-order Taylor expansion with reference to a known equilibrium state, Eq. (2) is transformed to a form appropriate for a linear incremental procedure.

$$\{\Delta\mathbf{p}\} = ([k] + [n_0] + [n_1] + [n_2])\{\Delta\mathbf{q}\} \quad (3)$$

where Δ denotes an incremental operator. The overall structural formulation is obtained by adding the formulations of individual elements.

In performing the incremental procedure, the increments of displacement vector can not be assumed. The loading pattern is, however, usually known and the increments of load vector can be applied. This requires an inverse of the stiffness matrix in each incremental step,

$$\{\Delta\mathbf{Q}\}_{\text{step } i+1} = ([K] + [N_0] + [N_1] + [N_2])_i^{-1}\{\Delta\mathbf{P}\}_{i+1} \quad (4)$$

where the capital letters are adopted to represent the assembled overall structural equations. Since the stiffness matrices thus derived are given in terms of the geometry of the undeformed elements, the application of Eq. (4) requires no coordinate transformation.

In analyzing the postbuckling response of plates, the membrane boundary conditions need to be considered in addition to the flexural boundary conditions. The latter are treated directly in solving Eq. (4), but the former must be treated separately, because of the coupling of membrane and flexure actions in the finite-deflection range. In order to treat the membrane edge conditions, the distribution of edge membrane stress and membrane displacement need to be determined. From the principle of statics, the stress-strain and strain-displacement equations are rewritten in a form where nodal membrane forces are expressed in terms of nodal degrees of freedom,

$$\{\mathbf{P_m}\} = [K_m]\{Q_m + \mathbf{Q_{0m}}\} + \{\mathbf{Q_f} + \mathbf{Q_{0f}}\}^T[F]\{\mathbf{Q_f} + \mathbf{Q_{0f}}\} \quad (5)$$

where the vector $\{\mathbf{P_m}\}$ denotes the nodal membrane forces; $\{\mathbf{Q_m}\}$ and $\{\mathbf{Q_f}\}$ are the vectors of nodal membrane and flexure displacements, respectively; $[K_m]$ is the membrane portion of the linear stiffness matrix; and $[F]$ relates the nodal membrane forces to the nodal flexure displacements.

At the beginning of each incremental step, Eq. (5) is solved by substituting the current nodal flexure displacements and imposing the membrane boundary conditions,

$$\{\mathbf{Q_m} + \mathbf{Q_{0m}}\} = [K_m]^{-1}(\{\mathbf{P_m}\} - \{\mathbf{Q_f} + \mathbf{Q_{0f}}\}^T[F]\{\mathbf{Q_f} + \mathbf{Q_{0f}}\}) \quad (6)$$

Having obtained the solution of flexure displacements $\{\mathbf{Q_f} + \mathbf{Q_{0f}}\}$ and membrane displacements $\{\mathbf{Q_m} + \mathbf{Q_{0m}}\}$ at the beginning of each incremental step, the distribution of nodal membrane stress is found by using Eq. (5). From these displacements, the incremental stiffness matrices in Eq. (4) are determined and the next incremental step can then be carried out.

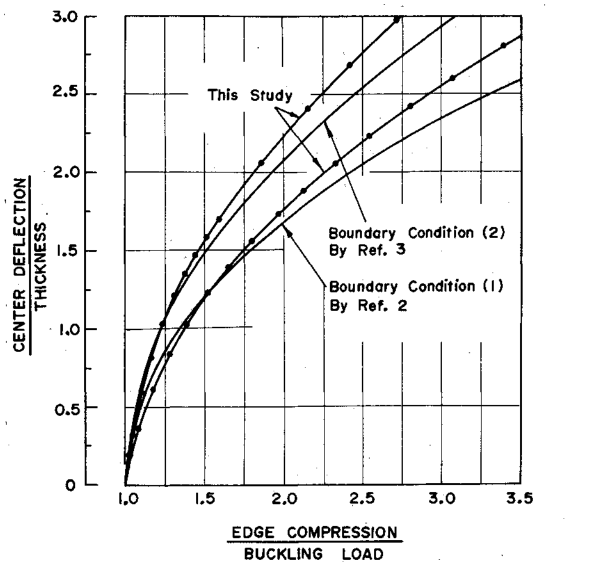

Fig. 1 Comparison of postbuckling center deflection of a square flat plate (Poisson's ratio = 0.316).

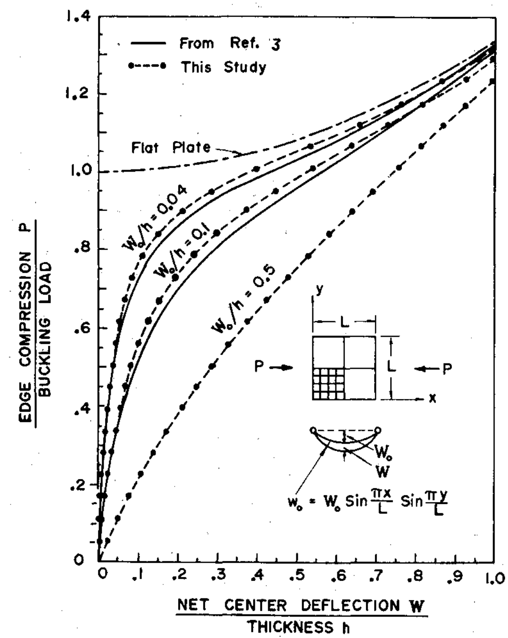

Fig. 2 Load-deflection of simply-supported square plates with initial deflections (Poisson's ratio = 0.316).

When a plate is subjected to in-plane loads, Eq. (4) is not directly applicable because of the intricate membrane-flexure coupling effect. However, Eq. (4) is directly solvable if only the transverse loads are applied. In each incremental step, the in-plane compressive load increments $\{\Delta\mathbf{P_x}\}$, $\{\Delta\mathbf{P_y}\}$, and $\{\Delta\mathbf{P_{xy}}\}$ are transformed into transverse load increment $\{\Delta\mathbf{P_z}\}$ by the form,

$$\{\Delta\mathbf{P_z}\} = \{\Delta\mathbf{P_x}\}\partial^2(w + w_0)/\partial x^2 + \{\Delta\mathbf{P_y}\}\partial^2(w + w_0)/\partial y^2 + 2\{\Delta\mathbf{P_{xy}}\}\partial^2(w + w_0)/\partial x\partial y \quad (7)$$

The transverse load increment $\{\Delta\mathbf{P_z}\}$ is then applied and Eq. (4) is solved for flexure displacements. The procedure described in the previous paragraph is thus followed. It is noted that when applying Eq. (7) to the first incremental step, the net deflection w is zero.

For the case of a flat plate with no initial curvature, the term w_0 vanishes, and Eq. (7) does not apply for the first step. In order to transform the in-plane compression into a lateral load, a slightly deflected shape must be initiated. Two disturbing techniques are suggested: 1) a distribution of a very small initial curvature is assumed to the plate; or 2) a distribution of a very small disturbing load is applied transversely to the plate. It is desirable that the distribution of the disturbing initial curvature or initial load be similar to the expected buckled shape; thus, for a single wave the curvature or load distribution is a half sine wave, but for a double wave postbuckled shape the load or curvature is distributed as a full sine wave. In the case of a flat plate, the bifurcation load is usually desired before the prediction of the postbuckling path. This load is found by solving the eigenvalue problem in which the determinant of the sum of matrices $[K]$ and $[N_1]$ vanishes.

To illustrate the potential of the method, numerical calculations of examples have been performed by using a conforming rectangular plate element. (The details are given in the full paper.) The first example chosen was a square plate subjected to an in-plane compression acting on two opposite edges. Two sets of membrane boundary condition were considered; in both sets the loaded edges of the plate were maintained straight with zero shearing stress, but in 1) the unloaded edges were maintained straight by a distribution of normal membrane stress, the resultant of which is zero, and the edges were allowed to move bodily in the plane of the plate and the shearing stress is zero, and in 2) the unloaded edges were free to wave in the plane of the plate and the normal membrane stress and shearing stress were zeros. The results for center deflection vs in-plane compression for both membrane boundary conditions are shown in Fig. 1. The solid circles describe the incremental step sizes used; sixteen discrete elements were used to idealize a quadrant of the plate. Alternative analytical results for this problem with two aforementioned membrane boundary conditions are available in Refs. 2 and 3, respectively. They are also shown in Fig. 1 for comparison. The agreement is fair. The agreement could be improved if 1) more terms are used in the series solutions in Refs. 1 and 2, and 2) more sophisticated membrane displacement, more incremental steps, and more elements are used in the present method.

The second example chosen was a square plate with various degrees of initial curvature. The edges were simply supported with membrane boundary condition (1). The results for prebuckling and postbuckling center deflection vs compression are shown in Fig. 2. For the cases that the maximum initial deflection equals to $0.04h$ and $0.1h$ (h is the thickness) alternative analytical results are available in Ref. 3. Good agreement is observed. In this figure, the results for the case where maximum initial deflection equals to $0.5h$ are not available from Ref. 3 for comparison.

From the results presented, it may be concluded that the ideas introduced here may provide a reasonable basis for the extension of discrete element method for the postbuckling problems.

References

[1] Levy, S., "Bending of Rectangular Plates with Large Deflections," TR 737, 1942, NACA.

[2] Coan, J. M., "Large-Deflection Theory for Plates with Small Initial Curvature Loaded in Edge Compression," *Journal of Applied Mechanics*, Vol. 18, No. 2, June, 1951, pp. 143–151.

[3] Hu, P. C., Lundquist, E. E., and Batdorf, S. B., "Effect of Small Deviations from Flatness on Effective Width and Buckling of Plates in Compression," TN 1124, Sept. 1946, NACA.